Lecture 14 30/06/2025 ①

Last time: Dirac operator $D^E = \sum_j c(e_j) \nabla^E_{e_j}$

Lichnerowicz Formula

$$(D^E)^2 = \Delta^E + \frac{r^X}{4} + \sum_{i<j} R^{E/S}(e_i, e_j) c(e_i) c(e_j)$$

VI. Atiyah-Singer index theorem for Dirac operator

VI.1) Statement of Atiyah-Singer index thm

Lemma: $(V, \langle , \rangle)$ Euclidean space of dim $m = 2k$

spinor space $S = \Lambda^\cdot \overline{W}^*$ ~~via~~ $V_{\mathbb{C}} = W \oplus \overline{W}$

Then $\forall\, a \in C(V)$

$$\mathrm{Tr}_s[c(a)] = \begin{cases} 0 & \text{if } a \in C^{m-1}(V) \\ 2^k & \text{if } a = i^k e_1 e_2 \cdots e_{2k-1} e_{2k} \\ & \{e_j\} \text{ oriented ONB} \end{cases}$$

Pf: $\tau = i^k c(e_1) \cdots c(e_{2k})$ on S

gives the $\mathbb{Z}_2$-grading $\tau = \pm 1$ on $S^\pm = \Lambda^\pm \overline{W}^*$

when orientation is given by $W \oplus \overline{W}$

$$\mathrm{Tr}_s[\tau] = \mathrm{Tr}[\tau^2] = \dim S = 2^k$$

Claim: $[C(V), C(V)] = C^{m-1}(V)$

For any $I \subset \{1, \cdots, m\}$ $I = \{i_1 < \cdots < i_k\}$

if $k \leq m-1$

we can suppose that $\ell \notin I$ $c(e_I) = c(e_{i_1}) \cdots c(e_{i_k})$

so $[c(e_\ell), c(e_\ell) c(e_I)]$

$= e_\ell^2 e_I - (-1)^{k+1} e_\ell e_I e_\ell$

$= -2 e_I$

②

$\Rightarrow \quad C^{m-1}(V) \subset [C(V), C(V)]$

$$\left\{\begin{array}{l} \mathrm{Trs}|_{[C(V),C(V)]} \equiv 0 \\ \mathrm{codim}\, C^{m-1}(V) = 1 \end{array}\right. \Rightarrow [C(V), C(V)] = C^{m-1}(V) \quad \#$$

$\dim V = m = 2k$

<u>Def</u> (Relative supertrace)

If $E = E^+ \oplus E^-$ is a $\mathbb{Z}_2$-graded $C(V)$-module

For $R \in \mathrm{End}_{C(V)}(E)$, define

$$\mathrm{Tr}_s^{E/S}[R] := 2^{-k}\, \mathrm{Tr}_s^{E}[\tau R]$$

Note that $\quad E = S \hat{\otimes} W \qquad E^{\pm} = (S \hat{\otimes} W)^{\pm}$

S: spinor space

$R \in \mathrm{End}_{C(V)}(E) \Rightarrow R \in \mathrm{End}(W)$ so $\mathrm{Tr}_s^{E/S}[R] = \mathrm{Tr}_s^{W}[R]$

<u>Def</u>: (E, h^E, ∇^E) $\mathbb{Z}_2$-graded Clifford module on X

∇^E: Hermitian Clifford connection

$$\mathrm{ch}_{E/S}(E, \nabla^E) := \mathrm{Tr}_s^{E/S}\left[e^{-\frac{\sqrt{-1}}{2\pi} R^{E/S}}\right] \in \Omega^{\mathrm{even}}(X, \mathbb{R})$$

closed diff. forms, coh. class independent of ∇^E

Theorem (Index Theorem for Dirac operator)

(X, g^{TX}) — oriented even-dimensional compact Riemannian manifold

(E, h^E, ∇^E) $\mathbb{Z}_2$-graded Clifford module.

$\mathrm{Ind}(D_+^E) = \dim \ker D_+^E - \dim \mathrm{coker}\, D_+^E$

Then we have

$$\mathrm{Ind}(D_+^E) = \int_X \underbrace{\hat{A}(TX, \nabla^{TX})\, ch^{E/S}(E, \nabla^E)}_{\text{closed diff. forms indep of } \nabla^{TX}, \nabla^E}$$

$$- \int_X \hat{A}(TX)\, ch^{E/S}(E).$$

Cor: If X is cpt spin manifold, and X has a Riemannian metric s.t. $r^X \geq 0$ with at least one pt $r^X(x_0) > 0$

Then

$$\int_X \hat{A}(TX) = 0$$

Pf: We take $E = S^{TX}$

$$ch^{E/S}(E, \nabla^E) = 1 \quad (W = \mathbb{C})$$

Lichnerowicz formula

$$(D^{S^{TX}})^2 = \Delta^{S^{TX}} + \frac{r^X}{4}$$

If $s \in \ker D^{S^{TX}}$

$$0 = \|D^{S^{TX}} s\|_{L^2}^2 = \langle (D^{S^{TX}})^2 s, s \rangle_{L^2}$$

$$= \langle \Delta^{S^{TX}} s, s \rangle_{L^2} + \frac{1}{4}\langle r^X s, s \rangle_{L^2}$$

$$= \langle \nabla^{S^{TX}} s, \nabla^{S^{TX}} s \rangle_{L^2} \geq 0$$

$$+ \frac{1}{4}\langle r^X s, s \rangle_{L^2} \geq 0$$

Since $r^X(x_0) > 0$

$\Rightarrow \quad s \equiv 0$ near x_0

$\nabla^{S^{TX}} s \equiv 0 \Rightarrow |s|^2_{h^{S^{TX}}}(x)$ is constant on X

$\Rightarrow \quad s \equiv 0$ on whole X

$\Rightarrow \quad \mathrm{Ind}(D_+^{S^{TX}}) = 0$ #

VI.2) Heat Kernel (General Theory)

Let (X, g^{TX}) be n-dim cpt oriented Riemannian manifold

$\rightsquigarrow$ dv_X Riemannian volume form

(E, h^E) Hermitian vector bundle on X

with ∇^E Hermitian connection

Recall :
$$\Delta^E = -\sum_j \left((\nabla^E_{e_j})^2 - \nabla^E_{\nabla^{TX}_{e_j} e_j} \right) \curvearrowright C^\infty(X, E)$$

Bochner Laplacian ≥ 0

Now take $Q \in C^\infty(X, \mathrm{End}(E))$ s.t. $Q(x)^* = Q(x)$ for $x \in X$

Assume that $Q(x) \geq -c \, \mathrm{Id}_E$ for some $c > 0$

Def : Generalized Laplace
$$H := \Delta^E + Q$$

After replacing Q by $Q + c\,\mathrm{Id}_E$, then we may always assume
$$H \geq 0$$

Properties :

① H is symmetric
$$\langle Hs, s' \rangle_{L^2} = \langle s, Hs' \rangle_{L^2}$$

② $\mathrm{Dom}(H) = H^2(X, E)$ Sobolev space
$$= \{ s \in L^2(X, E) : \nabla^E s \in L^2, \nabla^{T^*X \otimes E} \nabla^E s \in L^2 \}$$
$$\subset L^2(X, E)$$

Note that $C^\infty(X,E) \subset H^2(X,E) \underset{\text{dense}}{\subset} L^2(X,E)$

③ $H : \mathrm{Dom}(H) \subset L^2(X,E) \to L^2(X,E)$
is essentially self-adjoint.

Recall : results from PDE theory for elliptic operators.

$\nabla : C^\infty(X,E) \to C^\infty(X, T^*X \otimes E)$

$\nabla^k : C^\infty(X,E) \to C^\infty(X, T^*X^{\otimes k} \otimes E)$

induced by ∇^E & ∇^{T^*X} (Levi-Civita)

$W^k(X,E) = \{ s \in L^2(X,E) \text{ s.t. } \nabla^\ell s \in L^2 \text{ for } \ell = 1, \cdots, k \}$

$W^0(X,E) = L^2(X,E)$

$\supset W^1(X,E) \supset W^2(X,E) \supset \cdots \supset C^\infty(X,E)$

Sobolev spaces

Theorem X cpt manifold

① (Rellich) $\forall k \in \mathbb{N}$

$\mathrm{incl} : W^{k+1}(X,E) \hookrightarrow W^k(X,E)$

is a compact operator, that bded set of $W^{k+1}(X,E)$ is pre-cpt in $W^k(X,E)$

② (Sobolev) $\forall k \in \mathbb{N}$, $\ell > \frac{\dim X}{2} + k$ then

$W^\ell(X,E) \hookrightarrow C^k(X,E)$ continuous.

As a consequence $\bigcap_{\ell \in \mathbb{N}} W^\ell(X,E) = C^\infty(X,E)$

Thm (Elliptic estimates) X cpt, $\not{H}$ Laplacian

$\forall k \in \mathbb{N}$, $\exists C_k > 0$ s.t. $\forall s \in W^{k+2}(X, E)$

$$\|s\|_{W^{k+2}} \leq C_k \left(\|Hs\|_{W^k} + \|s\|_{L^2} \right)$$

(Gårding's inequality) $\forall s \in W^1(X, E)$

$$\|s\|^2_{W^1} \leq C\left(\langle Hs, s \rangle_{L^2} + \|s\|^2_{L^2} \right)$$

Thm (Regularity) X cpt. $\not{H}$ Laplacian

For $s \in L^2(X, E)$, if $\exists k \in \mathbb{N}$ s.t.

$$Hs \in W^k(X, E)$$

Then $s \in W^{k+2}(X, E)$

Cor: $(\not{H} + 1)^{-1} : L^2(X, E) \to L^2(X, E)$

is compact, self-adjoint, ≥ 0.

Pf: $1 + \not{H} : W^2(X, E) \to L^2(X, E)$

injective, since $1 + \not{H}$ is symmetric

$\operatorname{Im}(1 + \not{H})^{\perp} = \{0\}$

Im $(1+\not{H})$ is closed subspace of $L^2(X, E)$

$\rightsquigarrow (\not{H} + 1)^{-1} : L^2(X, E) \to W^2(X, E)$

bded linear op

Finally, by Rellich $W^2(X, E) \hookrightarrow L^2(X, E)$ is cpt

$\rightsquigarrow (\not{H} + 1)^{-1} : L^2(X, E) \to L^2(X, E)$ is cpt #

Prop (Spectral theory)

① $\exists$ complete orthonormal basis $\{\phi_i\}$ of $L^2(X,E)$ and $\mu_j > 0$ s.t.

$$(H+1)^{-1}\phi_j = \mu_j \phi_j \qquad \mu_j \searrow 0^+$$

② Moreover $0 < \mu_j \leq 1$, we define

$$\lambda_j = \frac{1}{\mu_j} - 1 \geq 0 \qquad \lambda_j \nearrow +\infty$$

$$\Rightarrow \begin{cases} H\phi_j = \lambda_j \phi_j & \forall j. \\ \phi_j \in C^\infty(X,E) \end{cases}$$

Proof: ① Using that $(H+1)^{-1} \circlearrowright L^2(X,E)$ is a self-adj. compact operator, $0 < (H+1)^{-1} \leq 1$

② $(H+1)^{-1}\phi_j = \mu_j \phi_j \subset W^2(X,E)$

$$\phi_j = \mu_j (H+1)\phi_j = \mu_j H\phi_j + \mu_j \phi_j$$

$$\Rightarrow \quad H\phi_j = \frac{1-\mu_j}{\mu_j}\phi_j = \lambda_j \phi_j$$

Moreover $(H-\lambda_j)\phi_j = 0$ and $0 \in C^\infty(X,E)$

$H-\lambda_j$ elliptic

$$\Rightarrow \phi_j \in C^\infty$$

#

(heat operator)

Def: For $t>0$, heat operator $e^{-tH}: L^2(X,E) \to L^2(X,E)$ is defined by

it is C^1 in $t>0$

and for any $s \in L^2(X,E)$

$$\begin{cases} \left(\frac{\partial}{\partial t} + H\right) e^{-tH} s = 0 \\ \lim\limits_{t\to 0} e^{-tH} s = s \quad \text{in } L^2(X,E). \end{cases} \qquad (*)$$

Rk: $(*)$ is the heat equation with initial condition $(t=0)$ $s \in L^2(X,E)$

Def (heat kernel)

Heat kernel $e^{-tH}(x,x')$ or $P_t(x,x')$, $x, x' \in X$, $t>0$,

is the Schwartz (integral) kernel of heat operator e^{-tH} on $L^2(X,E)$ with respect to Riemannian volume form dv on (X, g^{TX}), that is,

$\forall s \in C^\infty(X,E)$

$$(e^{-tH}s)(x) = \int_X e^{-tH}(x,x') s(x') dv(x')$$

where $e^{-tH}(x,x') \in E_x \otimes E^*_{x'}$

Rk: By heat equation, we have, given any $x' \in X$

$$\begin{cases} \left(\frac{\partial}{\partial t} + H_x\right) P_t(x,x') \equiv 0 \\ \lim\limits_{t\to 0} P_t(x,x') = \delta_{x'}(x) \quad \text{Dirac mass}, \end{cases}$$

$E \to X$

$E^* \to X$

consider $\pi_j : X \times X \longrightarrow X \qquad j = 1, 2$

$(x_1, x_2) \longmapsto x_j$

<u>Def</u>: $E \boxtimes E^* := \pi_1^* E \otimes \pi_2^* E^*$ a vector bundle over $X \times X$

<u>Thm</u>: $e^{-tH}(x, x')$ exists and is unique, ~~smooth~~ in $t > 0$, $(x, x') \in X \times X$

In particular

$t > 0$. $e^{-tH} \in C^\infty(X \times X, E \boxtimes E^*)$